Scalar-Tensor Models of Normal and Phantom Dark Energy

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Abstract

We consider the viability of dark energy (DE) models in the framework of the scalar-tensor theory of gravity, including the possibility to have a phantom DE at small redshifts z as admitted by supernova luminosity-distance data. For small z, the generic solution for these models is constructed in the form of a power series in z without any approximation. Necessary constraints for DE to be phantom today and to cross the phantom divide line $p = -\rho$ at small z are presented. Considering the Solar System constraints, we find for the post-Newtonian parameters that $\gamma_{PN} < 1$ and $\gamma_{PN,0} \approx 1$ for the model to be viable, and $\beta_{PN,0} > 1$ (but very close to 1) if the model has a significantly phantom DE today. However, prospects to establish the phantom behaviour of DE are much better with cosmological data than with Solar System experiments. Earlier obtained results for a Λ-dominated universe with the vanishing scalar field potential are extended to a more general DE equation of state confirming that the cosmological evolution of these models rule them out. Models of currently fantom DE which are viable for small z can be easily constructed with a constant potential; however, they generically become singular at some higher z. With a growing potential, viable models exist up to an arbitrary high redshift.

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1 Introduction

A major turning point in cosmology has been reached with the *observational* discovery that our Universe is accelerating now (and has been accelerating for several billion years in the past) [1, 2]. If interpreted in terms of the Einstein equations for the evolution of a Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological models with the (practically) zero spatial curvature (the latter follows from other arguments), this means that approximately two thirds of the total present energy density of matter in our Universe is due to some gravitationally unclustered component called Dark Energy (DE).

Observations of high redshift supernova, fluctuations of the cosmic microwave background (CMB) temperature and other effects tell us that the effective energy density ρ_{DE} of DE is very close to minus its effective pressure p_{DE} (see Eqs. (16,17) below for the exact, though conventional, definition of these quantities) and that they are both very weakly changing (if at all) with time and with the expansion of the Universe. The physical nature of DE is unknown at present, and three main logical possibilities exist (see [3, 4, 5, 6, 7] for reviews).

1. DE is a cosmological constant, as originally suggested by Einstein, with $\rho_{DE} = -p_{DE} = \Lambda/8\pi G = const$ exactly.[1]

2. Physical DE: DE is the energy density of some new, very weakly interacting physical field (e.g., a quintessence – a scalar field ϕ with some potential $V(\phi)$ minimally coupled to gravity).

3. Geometric DE: the Einstein general relativity (GR) equations are *not* the correct ones for gravity, but we write them in the Einsteinian form by convention, putting all arising additional terms into the r.h.s. of the equations and calling them the effective energy-momentum tensor of DE.

Of course, this classification is not absolute. In some cases, the difference between physical and geometric DE can, in turn, become conventional. E.g., for a non-minimally coupled scalar field (which constitutes a specific case of scalar-tensor gravity considered in this paper), equations of the model have the same form irrespective of the origin (physical or geometric) of this field. Another way of classifying DE models which is more invariant and, therefore, more important from the point of view of the observational determination of the type of DE is to divide these models into dynamical, if the DE description requires a new field degree of freedom ($\equiv$ a new particle from the quantum point of view), and non-dynamical in the opposite case. Then all physical DE models and many of the geometric ones belong to the first class, while the cosmological constant itself and some geometric models fall into the second category, i.e., the $F(R)$ model with the Palatini variation of its action where R is the scalar curvature.

At present, all existing observational data are in agreement with the simplest, first possibility (inside $\sim 2\sigma$ error bars in the worst case). The case of a cosmological constant is internally self-consistent and non-contradictive. The extreme smallness of the cosmological constant expressed in the either Planck, or even atomic units means

[1] $\hbar = c = 1$ is used throughout the paper.

only that its origin is *not* related to strong, electromagnetic and weak interactions (in particular, to the problem of the energy density of their vacuum fluctuations) at all. However, in this case we remain with one dimensionless constant only and can not say anything more (at least, at present). That is why the two other possibilities admitting a (slightly) variable dark energy have been also actively studied and compared with observational data recently. Moreover, properties of the present DE are remarkably *qualitatively* similar to those of an "early DE" that supported an inflationary stage in the early Universe. But in the latter case, we are sure that this early DE was unstable. So, it is natural to conjecture by analogy that the present DE is not stable, too.

On purely phenomenological grounds, one can consider DE models with a constant equation of state parameter $w_{DE} \equiv p_{DE}/\rho_{DE}$ different from -1. However, the latest observational data have already severely restricted this simplest alternative possibility to $|1 + w_{DE}| \lesssim 0.1$ (1σ error bars) [8, 9, 10]. Therefore, a viable alternative to the cosmological constant has to be looked for among more complicated models with $w_{DE} \neq const$. Such models include quintessence ones with different potentials $V(\phi)$ (see [3, 4, 5, 6] for numerous references), models with several minimally coupled scalar fields (see, e.g., [11]), those with direct non-gravitational coupling between DE and dark matter (DM) ([12] and following papers), unified models of DE and DM (the Chaplygin gas model [13] and others), etc. Then, however, it becomes very important to investigate if there exist models of DE for which a variable w_{DE} may cross the "phantom divide" line $w_{DE} = -1$ (we call DE "fantom" if it has $w_{DE}(z) < -1$ for a given z and "normal" in the opposite case). Note that the weak energy condition (WEC) is violated for phantom DE. This is not possible at all for quintessence models with the standard kinetic term and hardly possible, i.e., it requires non-generic initial conditions, for scalar field models having a non-standard kinetic term ("k-essence") [14] (see also [15]).

Indeed, analysis of the recent SNe data with redshifts up to $z = 1.7$ (the "Gold dataset" [16]) using fits containing at least 2 free parameters, e.g., the linear fit (76) for w_{DE} in terms of the scale factor a [17, 18] or the quadratic polynomial fit (77) for ρ_{DE} in terms of the redshift z [19], or with model independent methods, results in best fits to these data having a variable w_{DE} that steadily increases for redshifts $0 < z < 1$ and crosses the "phantom divide" somewhere between 0 and 0.5 [20, 21, 22, 23, 24, 25, 26, 27, 28, 29] (see, however, [30, 8, 31, 32] for a more conservative view – in the sense of returning back to a cosmological constant). This statement does not mean that an exact cosmological constant is excluded – it is still inside 1 or 2σ error bars, as was stated above. Moreover, one should be cautious with this result: it may be a consequence of trying to obtain a too fine-grained graph of $w_{DE}(z)$ from the luminosity distance $d_L(z)$ as has been already emphasized in [33]. In this respect, results for $w_{DE}(z)$ averaged over a range of redshifts > 0.3 (dubbed the "w-probe" in [34]) are more statistically reliable, e.g., the result that $\bar{w}_{DE} \approx -1.05^{+0.09}_{-0.07}$ obtained for the same Gold SNe dataset in [20] for $\Omega_{m,0} = 0.3$ and the averaging redshift interval $(0, 0.414)$. However, a remarkable possibility of crossing the phantom divide at recent redshifts, when DE has become the dominant

component of the Universe by its effective energy density, still remains viable. The latest Supernovae data with $z \sim 0.5$ [35] also admit $w_{DE}(z) < -1$ for $z < 0.5$, but not for larger redshifts, as a possible interpretation. Finally, the recent data on acoustic baryon oscillations in the present matter power spectrum (the Sakharov oscillations) [36, 37], while strongly restricting one direction in the plane of parameters for the two-parametric fits (76,77), leave the orthogonal direction practically unconstrained, therefore, permitting the recent phantom divide crossing (see Sec. 6).

So, if this striking behaviour of DE will be confirmed by future data, what is the best way to describe it? One possibility, a *ghost* phantom DE, i.e. a scalar field with the negative sign of its kinetic term, was first proposed in [38] and triggered a very large wave of publications (see, e.g., the recent papers [39, 40] for a list of references on this topic). However, it has been long known that theories of this type are plagued by quantum instabilities, the most dangerous of those being the process of creation of two particle+antiparticle pairs: one of the ghost field and another of any usual (non-ghost) field (see [41] for a recent investigation). Moreover, a ghost model of DE is unsatisfactory even at the classical level: it does not explain the observed large-scale isotropy and homogeneity of the Universe! Just the opposite, e.g., for the given Hubble constant H_0 averaged over angular directions in the sky, we would expect a universe to be very strongly anisotropic with the anisotropy energy density (i.e., the positive energy density of long-wavelength gravitational waves) being compensated by a large negative energy density of the ghost DE, or of a ghost component of DE in more complicated multi-component models of this type (e.g., in the two-field realization of the "quintom" model introduced in [42]). It is just a classical analogue of the quantum instability mentioned above, with the "usual" field being the gravitational one and with the dynamical quantum instability transformed into the problem of classical initial conditions. For this reason, we are sceptical regarding this approach as a whole.

Fortunately, there is no necessity to introduce a ghost field to explain possible phantom behaviour of DE including its phantom divide crossing. As was first emphasized in [43], scalar-tensor theories of gravity allow for this phenomenon. Scalar-tensor models of DE belong to the third class (geometric DE) according to the classification given above. This class of models is very rich and interesting. It contains the Einstein gravity plus a non-minimally coupled scalar field, as well as the higher-derivative $f(R)$ gravity, where R is the scalar curvature, as particular cases, besides allowing for DE phantom behaviour and transition to a normal DE. In this paper, we will concentrate on scalar-tensor DE models with the latter properties because of tentative observational evidence described above.[2] Note that a possible phantom behaviour of DE in scalar-tensor gravity has a conventional character. The reason is that the effective gravitational constants G_N and $G_{\rm eff}$ (see Sec. 2 below) are generically time-dependent while the definitions (16,17) of the DE effective energy density and pressure assume some constant G when writing the left-hand side of equations and splitting their right-hand side into energy-momentum tensors of non-relativistic

[2]The third possibility to get a phantom DE including the phantom divide crossing which is based on braneworld models [44] (see also the recent paper [45]) will not be discussed here.

matter (mainly non-baryonic cold dark matter) and dark energy. As a result, a part of the Einstein tensor $G_{\mu\nu}$ and the energy-momentum tensor of dark matter multiplied by a change in G_N is conventionally attributed to DE. In other words, phantom behaviour of scalar-tensor DE is always 'curvature-induced', in contrast to the ghost DE models or other models like those considered in [46, 47] where it may occur in the flat space-time already.

The possibility to get both a phantom DE and the phantom divide crossing in scalar-tensor gravity is related to the fact that this theory has two arbitrary and independent functions $F(\Phi)$ and $U(\Phi)$ (see the Lagrangian (1) below). Throughout this work we assume spatial flatness though this prior while well motivated theoretically can be challenged [48]. As has been shown in [43], two different types of observations, e.g., determination of the luminosity distance and the inhomogeneity growth factor in the non-relativistic component as functions of redshift, are necessary and sufficient for the total reconstruction of the microscopic Lagrangian (1) of scalar-tensor gravity. However, as shown in [49], a partial reconstruction using the luminosity distance data only could yield interesting information, too. In that case some assumption about one of the functions $F(\Phi)$ and $U(\Phi)$ has to be made, so that only one unknown function in the microscopic Lagrangian (1) remains to be found. Clearly, many different partial reconstruction strategies are possible and we explore some of them here with the aim to investigate which DE models in scalar-tensor gravity are viable.

It has been found in [49] that Λ-dominated universes with a vanishing potential U are ruled out as they lead to singular universes already at very low redshifts $z \sim 0.7$ (see also the recent paper [50]). Of course, completely regular but non-accelerating solutions do exist in this case. Among models with a non-zero potential U, one concrete example of a model with the phantom divide crossing was constructed in [50] where the functions $F(\Phi)$ and $U(\Phi)$ were given in the parametric form as functions of z up to $z \approx 3$. More examples for a non-minimally coupled scalar field ($F(\Phi) = F_0 - \xi\Phi^2$) with a non-zero potential U were investigated in [51], also for $z < 2$. In the present paper, we make a next step and extend these results in two directions: first, by constructing a generic solution for scalar-tensor DE models for $z < 1$ in the form of a power series in z; second, by investigating and numerically integrating some of these solutions up to large $z \gg 1$. The latter task appears to be necessary since, though DE is subdominant for $z \gg 1$ as compared to non-relativistic non-baryonic dark matter and baryons, its model itself may become intrinsically contradictory for large z, namely, F or $\dot{\Phi}^2$ may become negative for an unfortunate choice of $U(\Phi)$. It is also crucial to check that any DE model has the correct power-law behaviour for large z [52].

Finally, for the scalar-tensor theory of gravity, it is well known that the present value of $\frac{dF}{d\Phi}(z=0)$ is severely restricted from Solar System tests of post-Newtonian gravity (i.e. by the measured values of the post-Newtonian parameters). For this reason, there have been stated that the present phantom behaviour in scalar-tensor models of DE requires large amount of fine tuning and, thus, is unnatural [53, 54]. Therefore, it is important to investigate this problem in more detail to quantify what amount of fine tuning (and of what kind) is necessary for a significantly phantom behaviour of DE, and to determine the relation between this behaviour and results

of Solar System tests of gravity.

The paper is organized as follows. In Sec. 2 we define all quantities related to our scalar-tensor DE model and present the background evolution equations.

In Sec. 3 we derive the general integral solution for the quantity $F(z)$. In Sec. 4 we consider solutions in which DE scales as some power of the FLRW scale factor $a(t)$ and show their existence for DE of the phantom type (the latter requires a non-zero $U(\Phi)$).

In Sec. 5 we consider the small z behaviour of our model and find conditions for the violation of the WEC by DE today and for the phantom boundary crossing at small z. In Sec. 6 the general reconstruction of a microscopic model is considered and the observational constraint from the acoustic oscillations in the matter power spectrum is discussed. In Sec. 7 we consider the reconstruction for a constant potential more specifically and show that the model becomes singular at some redshift that cannot be arbitrarily high once we have chosen a specific equation of state w_{DE}. In Sec. 8 non-constant potentials are considered and a model which is asymptotically stable for large z is presented. Sec. 9 contains conclusions and discussion.

2 Background evolution

In this section we review the background evolution equations in a spatially flat FLWR universe. We consider a model where gravity is described by a scalar-tensor theory and we start with the following microscopic Lagrangian density in the Jordan frame

$$L = \frac{1}{2}\Big(F(\Phi)\ R - Z(\Phi)\ g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi\Big) - U(\Phi) + L_m(g_{\mu\nu})\ . \tag{1}$$

Since L_m is not coupled to Φ, the Jordan frame is the physical one. In particular, fermion masses are constant and atomic clocks measure the proper time t in it. The quantity $Z(\Phi)$ can be set to either 1 or -1 by a redefinition of the field Φ, apart from the exceptional case $Z(\Phi) \equiv 0$ when the scalar-tensor theory (1) reduces to the higher-derivative gravity theory $R + f(R)$. In the following, we will write all equations and quantities for the case $Z = 1$. For our purposes, L_m describes non-relativistic dust-like matter (baryons and cold dark matter) as we are interested in low redshift ($z \ll z_{eq}$) behaviour only. Here, z_{eq} denotes the equality redshift when the energy densities of non-relativistic matter and radiation are equal. In such a model, the effective Newton gravitational constant for homogeneous cosmological models is given by

$$G_N = (8\pi F)^{-1}. \tag{2}$$

As could be expected, G_N does not have the same physical meaning as in General Relativity, the effective gravitational constant $G_{\rm eff}$ for the attraction between two test masses is given by

$$G_{\rm eff} = G_N\ \frac{F + 2(dF/d\Phi)^2}{F + \frac{3}{2}(dF/d\Phi)^2}\ . \tag{3}$$

on all scales for which the field Φ is effectively massless [43] and $F > 0$. The condition $G_{\rm eff} > 0$ is one of the stability conditions of the scalar-tensor theory (1), it means

that the graviton is not a ghost. In fact, even at the purely classical level, it has been shown in [55] that a generic solution of (1) may not smoothly cross the boundary $G_{\rm eff} = 0$. Instead, a curvature singularity forms at this boundary which generic structure has been also constructed in [55]. This condition combined with another stability condition (see Eq. (10) below) results in $F > 0$, so $G_N > 0$, too.

We will write all equations in the Jordan frame using (1). Specializing to a spatially flat FLRW universe

$$ds^2 = -dt^2 + a^2(t)\ d\mathbf{x}^2\ , \tag{4}$$

the background equations are as follows:

$$3FH^2 = \rho_m + \frac{\dot\Phi^2}{2} + U - 3H\dot F\ , \tag{5}$$
$$-2F\dot H = \rho_m + \dot\Phi^2 + \ddot F - H\dot F\ . \tag{6}$$

The evolution equation of the scalar field Φ can be obtained from the two equations (5,6). Eliminating the quantity $\dot\Phi^2$ by combining these equations, we obtain a master equation for the quantity F which takes the following form when all quantities are expressed as functions of redshift z:

$$\begin{aligned} F'' &+ \left[(\ln h)' - \frac{4}{1+z}\right]\ F' + \left[\frac{6}{(1+z)^2} - \frac{2(\ln h)'}{1+z}\right]\ F \\ &= \frac{6u}{(1+z)^2\ h^2} F_0\ \Omega_{U,0} + 3\ (1+z)\ h^{-2} F_0\ \Omega_{m,0}\ , \end{aligned} \tag{7}$$

where a prime denotes the derivative with respect to z and we have introduced the quantities $h \equiv \frac{H}{H_0}$, $\Omega_{U,0} \equiv \frac{U_0}{3F_0\ H_0^2}$ and $u \equiv \frac{U}{U_0}$. The index 0 denotes the present moment here and below. The (dimensionless) relative energy density $\Omega_{m,0}$ is defined through $\Omega_{m,0} \equiv \frac{\rho_{m,0}}{3F_0 H_0^2}$. Once the master equation (7) is solved for F, we get the algebraic equation for $\Phi'(z)$:

$$\frac{\Phi'^2}{6} = -\frac{F'}{1+z} + \frac{F}{(1+z)^2} - \frac{F_0\ u}{(1+z)^2\ h^2}\ \Omega_{U,0} - \frac{F_0\ (1+z)}{h^2}\ \Omega_{m,0}\ . \tag{8}$$

The second stability condition of the scalar-tensor gravity (1) is

$$\omega_{BD} = \frac{F\Phi'^2}{F'^2} > -\frac{3}{2}\ , \tag{9}$$

where Φ'^2 is found from (8). Inequality (9) just expresses the positivity of the energy of the (helicity zero) scalar partner of the graviton, i.e. the positivity of the kinetic energy of the scalar field in the Einstein frame (see e.g. [49] for more details)

$$\phi'^2 \equiv \frac{3}{4}\left(\frac{F'}{F}\right)^2 + \frac{\Phi'^2}{2F} > 0 \tag{10}$$

with Φ'^2 taken from (8). With the $Z = 1$ parametrization, we cannot reconstruct the function $F(\Phi)$ when $-\frac{3}{2} < \omega_{BD} < 0$ since Φ'^2 becomes negative in this case. Indeed, these allowed negative values of ω_{BD} correspond to the parametrization choice $Z = -1$ in Eq.(1). The $Z = 1$ parametrization allows us to consider consistently only cases for which $\Phi'^2 \geq 0$, or equivalently $\omega_{BD} \geq 0$. However, the condition (9) with Φ'^2 given by (8) is the true condition that the theory is well behaved and it remains valid even for $\Phi'^2 < 0$. This is best understood in the Brans-Dicke parametrization $F = \Phi,\ Z = \frac{\omega_{BD}}{\Phi}$, where one can reconstruct the two functions $F > 0$ and $\omega_{BD} > -\frac{3}{2}$ from (7), (8), (9). For $\Phi'^2 \geq 0$ and $F \geq 0$, the inequality (9) is satisfied automatically. The inclusion of the range $-\frac{3}{2} < \omega_{BD} < 0$ is not of purely academic interest and can be important when one considers the reconstruction of DE models far in the past. We illustrate this with a specific example in Fig. 1.

Solar System experiments constrain the post-Newtonian parameters γ_{PN} and β_{PN} *today* (for these quantities, we drop here the subscript 0)

$$\gamma_{PN} = 1 - \frac{(dF/d\Phi)^2}{F + 2(dF/d\Phi)^2} \ , \tag{11}$$

$$\beta_{PN} = 1 + \frac{1}{4}\frac{F\ (dF/d\Phi)}{2F + 3(dF/d\Phi)^2}\frac{d\gamma}{d\Phi} \ , \tag{12}$$

as well as the quantity $\frac{\dot{G}_{\rm eff,0}}{G_{\rm eff,0}}$. The best present bounds are:

$$\begin{aligned} \gamma_{PN} - 1 &= (2.1 \pm 2.3) \cdot 10^{-5} \\ \beta_{PN} - 1 &= (0 \pm 1) \cdot 10^{-4} \\ \frac{\dot{G}_{\rm eff,0}}{G_{\rm eff,0}} &= (-0.2 \pm 0.5) \cdot 10^{-13}\ y^{-1} \ . \end{aligned} \tag{13}$$

where the first bound was obtained from the Cassini mission [56] and the other two from high precision ephemerides of planets [57] (the second bound has been recently confirmed by the Lunar Laser ranging [58] – their value is $\beta_{PN} - 1 = (1.2 \pm 1.1) \cdot 10^{-4}$).

As a consequence of the smallness of $\gamma_{PN} - 1$, the Brans-Dicke parameter ω_{BD} satisfies *today* the inequality

$$\omega_{BD,0} > 4 \times 10^4 \ . \tag{14}$$

The resulting bound on $F'(0)$ is very stringent

$$\frac{F'(0)}{\sqrt{F_0}} = \pm \frac{\Phi'(0)}{\sqrt{\omega_{BD,0}}} \ . \tag{15}$$

However, as was discussed in [43], the quantity ω_{BD} need not be so large as (14) in the past, though one can deduce a looser inequality applying up to redshift $z \lesssim 1$ with fairly reasonable assumptions. From (8, 15) we can derive the allowed range of initial values $F'(0)$ and we find $|F'(0)| \sim \omega_{BD,0}^{-\frac{1}{2}}$ and $\Omega_{DE,0} - \Omega_{U,0} > 0$, the result that we will recover in Sec. 5 when performing the small z expansion of all quantities. The peculiar case $F'(0) = 0$ together with $\Omega_{DE,0} - \Omega_{U,0} > 0$ corresponds to pure General Relativity today ($\omega_{BD} = \infty$).

As noted above, supernova observations permit DE to be of the phantom type, with the equation of state parameter $w_{DE} < -1$ at small redshifts. This is a strong motivation for considering DE models in the framework of the scalar-tensor theory of gravity. More generally, at present there is much interest in models with modified gravity of which the scalar-tensor theory is a well known representative. In scalar-tensor DE models, a meaningful definition of energy density and pressure of the DE sector requires some care (see also [53] for a detailed explanation). Let us *define* the energy density ρ_{DE} and the pressure p_{DE} in the following way:

$$3F_0 \; H^2 = \rho_m + \rho_{DE} \tag{16}$$
$$-2F_0 \; \dot{H} = \rho_m + \rho_{DE} + p_{DE} \; . \tag{17}$$

This just corresponds to the (conventional) representation of the true equation for scalar-tensor gravity interacting with matter in the *Einsteinian* form with the constant $G_0 = G_N(t_0)$:

$$R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} = 8\pi G_0 \left(T_{\mu\nu,m} + T_{\mu\nu,DE} \right) \; . \tag{18}$$

With these definitions, the usual conservation equation applies:

$$\dot{\rho}_{DE} = -3H(\rho_{DE} + p_{DE}) \; . \tag{19}$$

If we define the equation of state parameter w_{DE} through

$$w_{DE} \equiv \frac{p_{DE}}{\rho_{DE}} \; , \tag{20}$$

the time evolution of the DE sector is given by

$$\frac{\rho_{DE}(z)}{\rho_{DE,0}} \equiv \epsilon(z) = \exp\left[3 \int_0^z dz' \; \frac{1 + w(z')}{1 + z'}\right] \; . \tag{21}$$

Using (16,17), one gets

$$w_{DE} = \frac{\frac{2}{3}(1+z)\frac{d\ln H}{dz} - 1}{1 - \frac{H_0^2}{H^2} \; \Omega_{m,0}(1+z)^3} \; , \tag{22}$$

where

$$\Omega_m \equiv \frac{\rho_m}{3H^2 F_0} \; . \tag{23}$$

Then Eq.(16) can be rewritten as

$$h^2(z) = \Omega_{m,0} \; (1+z)^3 + \Omega_{DE,0} \; \epsilon(z) \tag{24}$$

where $\Omega_{DE,0} = 1 - \Omega_{m,0}$ by definition. The condition for DE to be of the phantom type, $w_{DE} < -1$, can be obtained from (22). It reads [3, 43]

$$\frac{dh^2}{dz} < 3 \; \Omega_m \; (1+z)^2 \; . \tag{25}$$

This inequality is modified in the presence of spatial curvature [48]. As was first emphasized in [43], the scalar-tensor gravity allow for phantom DE. Indeed, for these models

$$\rho_{DE} + p_{DE} = \dot{\Phi}^2 + \ddot{F} - H\dot{F} + 2(F - F_0)\ \dot{H}\ , \tag{26}$$

hence the weak energy condition for DE can be violated (see also [53]). Moreover, as it will be shown below, the weak energy condition may be violated even for the sum of DE and non-relativistic matter, i.e. for the whole right-hand side of Eqs. (18), leading to $dh/dz < 0$. However, such a strong violation corresponding to $w_0 \equiv w_{DE}(0) < (1 - \Omega_{m,0})^{-1} \approx -1.4$ is not supported by the existing data (though not completely excluded either).

The relation between the Hubble parameter and the luminosity distance in scalar-tensor gravity is the same as in GR [3]:

$$h^{-1}(z) = \left(\frac{d_L(z)}{1+z}\right)'\ ,\quad d_L(z) = H_0\ D_L(z)\ . \tag{27}$$

However, as discussed in [59, 60], when using supernova data to obtain $D_L(z)$, one has to take into account the dependence of the Chandrasekhar mass on $G_{\rm eff}$, so that the SnIa peak luminosity appears to be $\propto (G_{\rm eff}(z)/G_{{\rm eff},0})^{-3/2}$. As shown in [43], the full reconstruction of the functions $F(\Phi)$, $U(\Phi)$ requires two independent types of observations: $d_L(z)$ or another function of z that probes the background evolution, and the growth factor of matter perturbations $\delta\rho_m(z)/\rho_m(z)$ at some comoving scale much less than the Hubble scale, too. On the other hand, as emphasized in [49], one can already obtain powerful constraints from a partial reconstruction using $D_L(z)$ only. Such a partial reconstruction is possible when some additional condition is imposed on either F or U. This is the way we adopt in Sec. 7: we reconstruct the function F for a given Hubble parameter $H(z)$ and a constant potential U and investigate whether the resulting model is viable. Before embarking on such partial reconstructions, we derive first an integral form of the general solution of Eq. (7).

3 The master equation for F

Let us now consider the master equation (7) and present its general solution in an integral form. This will allow us to understand general properties of solutions, in particular, their dependence on initial conditions. The first step is to note that one solution of the homogeneous equation (7) (without source term) is given by

$$F \propto (1+z)^2\ . \tag{28}$$

This suggests us, following [49], to introduce the function f defined as follows

$$\frac{F(z)}{F_0} \equiv (1+z)^2 f(1+z)\ , \tag{29}$$

[3] Note that the definitions of $d_L(z)$ and $D_L(z)$ are interchanged in [17] and [48].

in terms of which Eq.(7) becomes

$$f'' + (\ln h)' f' = \frac{6\ u}{x^4\ h^2}\ \Omega_{U,0} + \frac{3}{x\ h^2}\ \Omega_{m,0}\ , \tag{30}$$

where we have introduced the variable $x \equiv 1+z$. Due to the absence of any term proportional to f, in accordance with (28,29), Eq. (30) is easily integrated formally, and its general solution has the integral form

$$\begin{aligned} f(x) = 1 + \left[-2 + \frac{F'(x=1)}{F_0}\right]\ \frac{d_L}{x}\ &+\ 6\ \Omega_{U,0} \int_1^x \frac{dx'}{h(x')} \int_1^{x'} \frac{u(x'')\ dx''}{x''^4\ h(x'')} \\ &+\ 3\ \Omega_{m,0} \int_1^x \frac{dx'}{h(x')} \int_1^{x'} \frac{dx''}{x''\ h(x'')}\ . \end{aligned} \tag{31}$$

A nontrivial dependence on initial conditions is through the second term only. We see that for a given initial value $F'(z=0)$, this term is proportional to the dimensionless luminosity distance d_L. Due to Eq. (15), we have $f'(x=1) = -2 \pm \frac{|\Phi'(z=0)|}{\sqrt{F_0\ \omega_{BD,0}}} \approx -2$, indicating that we must start today, on observational grounds, very close to GR. Hence, the second term of the general solution is bound to be negative and it is this term that will possibly push the quantity $f(z)$ downwards for increasing z. Finally, the corresponding quantity F is trivially obtained using (29). If we *choose* U to be constant, we can implement the reconstruction of the microscopic Lagrangian. By inspection of (7) for constant U, we find the asymptotic solution

$$F(x) = C_1\ x^2 + C_2\ x^{\frac{3}{2}} + F_0\ , \tag{32}$$

in complete agreement with (31) setting $u=1$. We can now proceed with the general reconstruction scheme [43, 49] and find $\Phi(z)$, and hence $z(\Phi)$, by integration of (8). This would finally give us $F(\Phi)$.

Let us come back to the solution (28). It corresponds to the parametrization $Z=-1$ with

$$F(\Phi) = \frac{1}{6}\Phi^2\ , \quad \omega_{BD} = -\frac{3}{2}\ , \tag{33}$$

where Φ is defined up to some constant. From (33) it is seen that this solution is unphysical in view of (9), see [49]. It is interesting that (33) corresponds to a conformally coupled scalar field in the Jordan frame satisfying

$$\ddot{\Phi} + 3H\dot{\Phi} + \frac{R}{6}\Phi = 0\ . \tag{34}$$

4 Scaling solutions

Let us now consider the so-called scaling solutions for which DE scales as some power of a,

$$\rho_{DE} \propto a^{-3\gamma}\ , \tag{35}$$

which attracted a lot of interest previously.[4] Clearly, for these solutions DE has an effective barotropic (constant) equation of state, viz.

$$w_{DE} = -1 + \gamma \ , \tag{36}$$

and we can readily write formally the general scaling solution by substituting

$$h^2 = \Omega_{m,0}\ (1+z)^3 + \Omega_{DE,0}\ (1+z)^{3\gamma} \ , \tag{37}$$

into the integral expression (31). We emphasize that this gives the general scaling solution irrespective of any limiting case and for an arbitrary potential, the only assumption is that of the spatial flatness. Using (31) and (37), general analytic expressions for $F(z)$ can be obtained only for the cases $\gamma = 1,\ 4/3$ (see Appendix).

In analogy with a minimally coupled scalar field (quintessence), a scaling solution satisfies

$$\frac{\rho_{DE} + p_{DE}}{\rho_{DE}} = 1 + w_{DE} = \gamma \ . \tag{38}$$

However, in contrast to the minimally coupled scalar field case with a positive potential, (38) does not imply $0 \le \gamma \le 2$, in particular γ can be negative which corresponds to phantom DE.

To get insight into the ability of scalar-tensor DE models to produce various equation of state parameters w_{DE}, it is instructive to study first scaling solutions in the absence of dust-like matter ($\rho_m = 0$). In this way, a lower limit on w_{DE} for realistic solutions with $\rho_m \neq 0$ can be obtained. These scaling solutions can also be considered as describing the asymptotic future of our universe when $\Omega_m \to 0$ and $\Omega_{DE} \to 1$. Then $a(t) \propto |t|^q$ with $q = \frac{2}{3\gamma}$. For phantom (or, super-inflationary) solutions, $q < 0$ and then the moment $t = 0$ corresponds to the 'Big Rip' singularity [38, 62].

It is straightforward to check that such scaling solutions can exist only when $F = \alpha\Phi^2$ with $\alpha = const$. This form of F corresponds to a constant Brans-Dicke parameter $\omega_{BD} = \frac{1}{4\alpha}$ for $Z(\Phi) = 1$. Further on, we assume that $\alpha > 0$ for stability of the theory. The two scaling solutions in the vanishing potential case (the pure Brans-Dicke theory) are [63]

$$|\Phi| \propto |t|^r, \quad r = \frac{1 \pm 3\sqrt{1 + \frac{1}{6\alpha}}}{8\left(1 + \frac{3}{16\alpha}\right)} \ , \quad q = \frac{1 + \frac{1}{4\alpha} \mp \sqrt{1 + \frac{1}{6\alpha}}}{4\left(1 + \frac{3}{16\alpha}\right)} > 0 \ . \tag{39}$$

Hence it is not possible to get a scaling solution with $w_{DE} < -1$ with vanishing (or negligible) potential $U(\Phi)$. We note that the above solution reduces to (28,33) obtained for $Z = -1$ when we formally put $\alpha = -\frac{1}{6}$ in (39). Indeed we get $q = 1,\ r = -1$, hence $F \propto a^{-2}$ which is exactly the solution (28).

However, scaling solutions supported by both the kinetic and the potential energy of Φ can exist in the presence of a polynomial potential $U(\Phi) = U_0|\Phi|^n$ (we assume

[4] Our definition for ρ_{DE}, Eq.(16), differs from that used in [61], and this explains why our results differ from theirs.

that the potential is positive, so $U_0 > 0$). We have for these solutions

$$r = \frac{2}{2-n} \ , \quad q = \frac{2\left(n+2+\frac{1}{\alpha}\right)}{(n-2)(n-4)} \tag{40}$$

while the inequality $q^2 + 2qr > \frac{r^2}{6\alpha}$ has to be satisfied (this is always the case if $\alpha \ll 1$ and $n = \mathcal{O}(1)$). They were first found in [64] (see also the analysis of their stability in [65, 66]) but they have not been discussed yet in connection with phantom DE. Note that there is no phantom behaviour for usually considered quadratic and quartic potentials.

So, we see that it is possible to get scaling solutions with $q < 0$, $w_{DE} < -1$ if $2 < n < 4$. For these solutions, $r < 0$, too, so that $a(t)$ diverges in the "Big Rip" singularity at some finite moment of time in future. It is clear that, relaxing the requirement of scaling behaviour, it is possible to add some amount of dust-like matter to these solutions while still keeping the phantom behaviour of DE. However, the amount of 'fantomness' exhibited by them, i.e. the modulus of the minimal possible value of $w_{DE} + 1 = 2/3q$, is very small for small values of α (equivalently large ω_{BD}), viz.

$$w_{DE} + 1 \geq -\frac{\alpha}{3} \ , \tag{41}$$

where the equality is achieved for $n = 3$. Thus, the conclusion is that polynomial potentials and scaling solutions in viable scalar-tensor DE models can lead to violation of the WEC, however, by the small amount $\sim \omega_{BD}^{-1}$ only.

5 Small z expansion and Solar System gravity tests

Of course, scaling solutions considered in the previous section are very specific ones. Let us now study generic solutions describing DE in the scalar-tensor gravity. Since, as was explained in the Introduction, if DE crosses the phantom boundary at all, it has done it in a very recent epoch, at small z, it is natural to study the expansion of a generic solution in powers of redshift z. For each solution $H(z)$, $\Phi(z)$, the basic microscopic functions $F(\Phi)$ and $U(\Phi)$ can be expressed as functions of z and expanded into Taylor series in z:

$$\frac{F(z)}{F_0} = 1 + F_1\ z + F_2\ z^2 + ... > 0 \ , \tag{42}$$

$$\frac{U(z)}{3F_0\ H_0^2} \equiv \Omega_{U,0} u = \Omega_{U,0} + u_1\ z + u_2\ z^2 + ... \ . \tag{43}$$

Note that this expansion produces two parameters in each order of z which are independent of initial conditions and can be expressed through derivatives of $F(\Phi)$ and

$U(\Phi)$ with respect to Φ. The corresponding expansion for $\Phi'(z)$ is:

$$
\begin{aligned}
(F_0)^{-\frac{1}{2}}\Phi'(z) &= \Phi_0' + \Phi_1'\ z + \Phi_2'\ z^2 + ... \\
&= \Delta + \frac{1}{\Delta}\Bigg[6(F_1 - F_2 + \Omega_{U,0} - 1) - 3(\Omega_{m,0} + u_1) \\
&+ \frac{3(\Omega_{m,0} + \Omega_{U,0})}{\frac{F_1}{2} - 1}(4F_1 - 2F_2 + 6\Omega_{U,0} + 3\Omega_{m,0} - 6)\Bigg] \cdot z + \ldots
\end{aligned} \tag{44}
$$

with $\Delta^2 \equiv 6\ (\Omega_{DE,0} - \Omega_{U,0} - F_1)$. As we will see below, $\Delta^2 > 0$. In principle, the expansion (44) can be inverted to get $z(\Phi)$. From (42,43), all other expansions can be derived:

$$
\begin{aligned}
h^2(z) &= 1 + h_1\ z + h_2\ z^2 + ...\ , &(45)\\
\epsilon(z) &= 1 + \epsilon_1\ z + \epsilon_2\ z^2 + ... > 0\ , &(46)\\
w_{DE}(z) &= w_0 + w_1\ z + w_2\ z^2 + ...\ , &(47)\\
H_0^{-1}\frac{\dot{G}_{\rm eff}}{G_{\rm eff}} &= g_0 + g_1\ z + g_2\ z^2 + ..., &(48)
\end{aligned}
$$

which can be used in order to constrain parameters of our model.

There are two types of observational constraints at small redshifts. The first of them includes those ones that follow from Solar System and other tests of possible deviations of the scalar-tensor gravity from GR at the present moment ($z = 0$); in particular, $|g_0| \lesssim 10^{-3}$ at the 2σ confidence level from the last of Eqs. (13). Other constraints follow from cosmological tests and refer to the whole range of redshifts up to $z \sim 1$ and higher redshifts, depending on the nature of the test. In particular, if we assume that the present supernova data admit (or even favour) $w_{DE}(z) < -1$ for $z \lesssim 0.3$ as was argued in the Introduction, then $\frac{d\epsilon}{dz} = \epsilon_1 + 2\ \epsilon_2\ z + 3\ \epsilon_3\ z^2 + ... < 0$ for $z \lesssim 0.3$. In particular, we must have $\epsilon_1 < 0$ that is equivalent to Eq. (25) taken at $z = 0$.

It follows from the substitution of these expansions into Eqs. (7,8) that

$$
\begin{aligned}
h_1 &= \frac{1}{1-\frac{F_1}{2}}\Big(6 - 3\Omega_{m,0} - 6\Omega_{U,0} - 4F_1 + 2F_2\Big)\ , \\
h_2 &= \frac{3}{(\frac{F_1}{2}-1)^2}\Bigg[F_1\left(\frac{5}{2}F_1 - 3F_2 - \frac{F_3}{2} + 4\Omega_{U,0} + \frac{u_1}{2} + \frac{11}{4}\Omega_{m,0} - 7\right) \\
&\quad + F_2^2 - 3F_2\Omega_{U,0} - \frac{3}{2}F_2\Omega_{m,0} + 4F_2 + F_3 - 5\Omega_{U,0} - u_1 - 4\Omega_{m,0} + 5\Bigg]\ , \qquad (49) \\
1 + w_0 &= \frac{\frac{5}{2}F_1 - 2F_2 - 6(\Omega_{DE,0} - \Omega_{U,0}) + \frac{3}{2}F_1\ \Omega_{DE,0}}{3\Omega_{DE,0}(\frac{F_1}{2}-1)}\ , \qquad (50) \\
w_1 &= \frac{1}{3\Omega_{DE,0}}\Bigg[\frac{1+6\frac{\Omega_{m0}}{\Omega_{DE,0}}}{\frac{F_1}{2}-1}\Big(4F_1 - 2F_2 + 6\Omega_{U,0} - 3\Omega_{DE,0} - 3\Big) - 9\frac{\Omega_{m0}}{\Omega_{DE,0}} \\
&+ \frac{6}{(\frac{F_1}{2}-1)^2}\Bigg(\frac{5}{2}F_1^2 - 3F_1F_2 - \frac{F_1F_3}{2} + 4F_1\Omega_{U,0} + \frac{F_1u_1}{2} + \frac{11}{4}F_1\Omega_{m0} - 7F_1 \\
&+ F_2^2 - 3F_2\Omega_{U,0} - \frac{3}{2}F_2\Omega_{m0} + 4F_2 + F_3 - 5\Omega_{U,0} - u_1 + 4\Omega_{DE,0} + 1\Bigg)\Bigg] \\
&- \frac{1}{3\Omega_{DE,0}^2(\frac{F_1}{2}-1)^2}\Big(4F_1 - 2F_2 + 6\Omega_{U,0} - 3\Omega_{DE,0} - 3\Big)^2\ . \qquad (51)
\end{aligned}
$$

The quantities $F_1,\ F_2,\ \Omega_{DE,0} - \Omega_{U,0}$ satisfy important constraints. For $\omega_{BD,0}$, we have the expression

$$\omega_{BD,0} = \frac{6(\Omega_{DE,0} - \Omega_{U,0} - F_1)}{F_1^2} = \frac{\Delta^2}{F_1^2} \qquad (52)$$

which should be very large and positive, see (14). Therefore, we must have $|F_1| \ll 1$ and $\Delta^2 \approx 6(\Omega_{DE,0} - \Omega_{U,0}) > 0$.[5] Moreover, since $\Delta^2 < 6\Omega_{DE,0} < 5$ for positive U,

$$|F_1| < \left(\frac{5}{\omega_{BD,0}}\right)^{1/2} \lesssim 10^{-2}\ . \qquad (53)$$

Thus, two cases are possible. In the first case, the further coefficients F_2, F_3, etc. in the expansion (42) are all of the order of F_1, i.e. they satisfy the inequality (53), too. In this case, the first derivative of F with respect to z or Φ at the present moment is not atypical compared to other derivatives. Then, however, a possible amount of phantomness (the quantity defined in the end of the previous section) is also of the order of $|F_1|$, i.e. less than 1%. Such DE will be practically indistinguishable from a cosmological constant.

Another possibility which admits 'significant phantomness', namely, $\min(1+w_{DE}(z)) < -0.01$, takes place if $|F_2|,\ |F_3|,\ (\Omega_{DE,0} - \Omega_{U,0})$ and so on are significantly larger than

[5] Note that this condition is not satisfied in the recent paper [67] that results in unphysical nature of its best-fit solution for $z < 0.2$.

$|F_1|$. It is a matter of taste if one considers the second possibility as 'fine tuned' with respect to the first one; observations should finally tell us if significant phantomness does exist or not. In any case, it is clear that if we are interested in any prediction for a significant deviation from the cosmological constant variant of DE, we may neglect F_1 as compared to F_2 and other parameters (but not in those expressions where it enters as a multiplier).

For $|F_1| \ll 1$, all the expansions above simplify significantly, and we have, in particular,

$$1 + w_0 \simeq \frac{2F_2 + 6(\Omega_{DE,0} - \Omega_{U,0})}{3\Omega_{DE,0}} . \tag{54}$$

From (54), the necessary condition to have phantom DE *today* reads

$$\left(\frac{d^2F}{d\Phi^2}\right)_0 = \frac{F_2}{3\,(\Omega_{DE,0} - \Omega_{U,0})} < -1 . \tag{55}$$

In particular, $F_2 < 0$, because $\Omega_{DE,0} - \Omega_{U,0} > 0$ as discussed above.[6] In the same limit, $h_1 < 0$, i.e. the WEC is violated for the total effective energy-momentum tensor for matter + DE (for the whole right-hand side of Eq. (18)), if the stronger inequality is fulfilled:

$$F_2 < -\frac{3}{2}(1 + \Omega_{DE,0} - 2\Omega_{U,0}) . \tag{56}$$

The expression for w_1 becomes in the limit $|F_1| \ll 1$

$$\begin{aligned} w_1 &= \frac{1}{3\Omega_{DE,0}}\left[\left(1 + 6\frac{\Omega_{m,0}}{\Omega_{DE,0}}\right)(2F_2 + 3 + 3\Omega_{DE,0} - 6\Omega_{U,0}) - 9\frac{\Omega_{m,0}}{\Omega_{DE,0}}\right] \\ &+ \frac{2}{\Omega_{DE,0}}\left[F^2\left(F_2 - 3\Omega_{U,0} - \frac{3}{2}\Omega_{m,0} + 4\right) - 5\Omega_{U,0} + 4\Omega_{DE,0} + 1 + F_3 - u_1\right] \\ &- \frac{1}{3\Omega_{DE,0}^2}(2F_2 + 3\Omega_{DE,0} + 3 - 6\Omega_{U,0})^2 \qquad (57) \\ &= \frac{9(w_0-1)^2\Omega_{DE,0} - 8 - 6w_0^2 + 9\Omega_{m,0} - 18\Omega_{U,0} + w_0\,(26 - 9\Omega_{m,0} + 18\Omega_{U,0})}{2} \\ &+ \frac{(-6w_0 + 14\Omega_{U,0} + 3\Omega_{m,0}(-3\Omega_{U,0} + 2w_0))}{\Omega_{DE,0}} + \frac{2}{\Omega_{DE,0}}\,(F_3 - u_1) , \qquad (58) \end{aligned}$$

where the last equality (58) is obtained using (54). Since F_3 and u_1 are free parameters determined by a concrete choice of $F(\Phi)$ and $U(\Phi)$, it is well possible to have $w_1 > 0$ for $w_0 < -1$ in order to realize a smooth phantom divide crossing at small z for DE in the scalar-tensor gravity.

[6] In this place our considerations intersect with those of the recent paper [68]. The condition (55) corresponds to $\beta_0 > 1$ in the notation of that paper. The other case mentioned there, $\beta_0 \sim -1$, does not lead to significant fantomness of DE with our definition of the DE energy-momentum tensor.

We obtain further

$$
\begin{aligned}
g_0 \;=\; & F_1 \Bigg\{ 1 - \frac{1}{(\frac{F_1}{2}-1)(F_1^2-3F_1-3\Omega_{U,0}+3\Omega_{DE,0})(F_1^2-4F_1-4\Omega_{U,0}+4\Omega_{DE,0})} \\
& \times \Bigg[\Bigg((F_1^2-4F_2)(\frac{F_1}{2}-1) - (5F_1-2F_2+6\Omega_{U,0}-3\Omega_{DE,0}-5)F_1 \Bigg) \\
& \times (F_1+\Omega_{U,0}-\Omega_{DE,0}) + (4F_1-2F_2+6\Omega_{U,0}-3\Omega_{DE,0}-3)F_1^2 \\
& + (F_1F_2 - \frac{5}{2}F_1 + \frac{F_1u_1}{2} - \frac{3}{2}F_1\Omega_{DE,0} - 6\Omega_{U,0} - u_1 + 6\Omega_{DE,0})F_1 \Bigg] \Bigg\} \; . \qquad (59)
\end{aligned}
$$

Expression (59) simplifies considerably for $|F_1| \ll 1$:

$$
g_0 \simeq F_1 \left(1 - \frac{F_2}{3(\Omega_{DE,0}-\Omega_{U,0})}\right) = F_1 \left(1 - \left(\frac{d^2F}{d\Phi^2}\right)_0\right) \; . \qquad (60)
$$

Note that g_0 and F_1 have the same sign for the case of (significantly) fantom DE. Finally, for the post-Newtonian parameter β_{PN} and γ_{PN} we have

$$
\beta_{PN} \;=\; 1 + \frac{\omega'_{BD}}{4(2\omega_{BD}+3)(\omega_{BD}+2)^2}\frac{F}{F'} \qquad (61)
$$

$$
\gamma_{PN} \;=\; 1 - \frac{1}{\omega_{BD}+2} \; . \qquad (62)
$$

Taking the zeroth order in z, we obtain :

$$
\beta_{PN,0} \;=\; 1 + \frac{\omega_{BD,1}}{4(2\omega_{BD,0}+3)(\omega_{BD,0}+2)^2}\frac{1}{F_1} \; . \qquad (63)
$$

$$
\gamma_{PN,0} \;=\; 1 - \frac{1}{\omega_{BD,0}+2} \; . \qquad (64)
$$

For $|F_1| \ll 1$ and $|F_1 u_1/\Delta^2| \ll 1$, we have:

$$
\omega_{BD,0} \;\simeq\; 6(\Omega_{DE,0}-\Omega_{U,0})\,\frac{1}{F_1^2} \qquad (65)
$$

$$
\omega_{BD,1} \;\simeq\; 24(\Omega_{DE,0}-\Omega_{U,0})\,\frac{F_2}{F_1^3} \; , \qquad (66)
$$

which finally yields

$$
\gamma_{PN,0} = 1 - \frac{F_1^2}{6(\Omega_{DE,0}-\Omega_{U,0})} \; , \quad \beta_{PN,0} = 1 - \frac{F_1^2\;F_2}{72(\Omega_{DE,0}-\Omega_{U,0})^2} \; . \qquad (67)
$$

Now we can use (50,52,60,67) to extract information from the Solar System constraints. First we note that all the Solar System constraints (14),(13) are satisfied for a sufficiently small $|F_1|$ (less than the upper bound (53)) independently of any requirement concerning the present DE equation of state, in particular, whether one

has today DE of the phantom type or not. From (62), it is seen that $\gamma_{PN,0} \approx 1$ and $\gamma_{PN} < 1$ for any viable scalar-tensor model of DE having $\omega_{BD,0}$ positive and large. For significantly phantom DE, other constraints follow from (67):

$$\beta_{PN,0} > 1 \ , \tag{68}$$

$$\frac{\gamma_{PN,0}-1}{\beta_{PN,0}-1} = \frac{12\,(\Omega_{DE,0}-\Omega_{U,0})}{F_2} = 6\,\Omega_{DE,0}\,\frac{1+w_0}{F_2} - 4 \ . \tag{69}$$

Therefore, for significantly phantom DE

$$-4 < \frac{\gamma_{PN,0}-1}{\beta_{PN,0}-1} < 0 \ . \tag{70}$$

It is possible to invert formulas (60,67) and express F_1, F_2 and $\Omega_{DE,0} - \Omega_{U,0}$ through the Solar System observables $\gamma_{PN,0}$, $\beta_{PN,0}$ and g_0:

$$F_1 = g_0\,\frac{\gamma-1}{\gamma-1-4(\beta-1)} \tag{71}$$

$$F_2 = -2\,g_0^2\,\frac{\beta-1}{[\gamma-1-4(\beta-1)]^2} \tag{72}$$

$$\Omega_{DE,0}-\Omega_{U,0} = -\frac{1}{6}\,g_0^2\,\frac{\gamma-1}{[\gamma-1-4(\beta-1)]^2} \tag{73}$$

$$1+w_0 = -\frac{1}{3}\,g_0^2\,\frac{4(\beta-1)+\gamma-1}{\Omega_{DE,0}\,[\gamma-1-4(\beta-1)]^2} \tag{74}$$

Thus, in principle, it is possible to test $w_0 < -1$ in the Solar System as was recently discussed in [68]. However, this may be very difficult to perform in practice since the small parameter F_1 enters quadratically into $\gamma_{PN,0} - 1$ and $\beta_{PN,0} - 1$ while not appearing (in some limit) in $1 + w_0$. So, a rather significant fantomness of DE is typically accompanied by very small deviations of the post-Newtonian parameters from their GR values. E.g., let us take $\Omega_{DE,0} = 0.7$, $\Omega_{U,0} = 0.6$, $\gamma_{PN,0} - 1 = -2 \cdot 10^{-5}$ (the latter being marginally possible at the 2σ level) and $w_0 = -1.2$. Then, from Eqs. (67,54), we get $|F_1| = 3.5 \cdot 10^{-3}$, $F_2 = -0.51$ and $\beta_{PN,0} - 1 = 0.85 \cdot 10^{-5}$ – an order of magnitude below the present upper limit.[7] Further, $w_1 = 2.9\,(F_3 - u_1) - 0.70$. Finally, in this case $|g_0| = 0.95 \cdot 10^{-2}$ that is an order of magnitude larger than the 2σ upper limit following from the last of Eqs. (13) ! So, if this upper bound will be confirmed, $|F_1|$ has to be decreased by an order of magnitude which results in $\beta_{PN,0} - 1$ and $\gamma_{PN,0} - 1$ being on the level of 10^{-7}. For comparison, in the extreme opposite case $\Omega_{U,0} = 0$ with the same values of $\Omega_{DE,0}$, $\gamma_{PN,0}$ and w_0, we get $|F_1| = 9.2 \cdot 10^{-3}$, $F_2 = -2.3$, $\beta_{PN,0} - 1 = 0.55 \cdot 10^{-5}$, $w_1 = 2.9\,(F_3 - u_1) + 1.5$ and even larger $|g_0| = 1.9 \cdot 10^{-2}$.

This shows also that the measurement of $\dot{G}_{\rm eff,0}/G_{\rm eff,0}$ is the most critical among Solar System tests of scalar-tensor DE since this quantity is proportional to the first power of the small parameter $|F_1|$ (apart from the exceptional case $(d^2F/d\Phi^2)_0 = 1$

[7] Let us emphasize once more that the formulas (60,67) were obtained under the assumption $|F_2| \gg |F_1|$. For this reason, they are not applicable, e.g., to the scaling solution (40) for which $F_2 = -F_1/2 > 0$, $\gamma_{PN} = 1 - 4\alpha$ in the limit $\alpha \ll 1$, and $\beta_{PN} \equiv 1$.

which does not lead to the present phantom behaviour of DE). Also, to determine w_1 which is necessary in order to consider the possibility of phantom boundary crossing, the determination of $(d^2 \ln G_{\text{eff}}/dt^2)_0$ is required, something that is hardly possible. Thus, testing the phantom behaviour of scalar-tensor DE in the Solar System may be much more difficult than in cosmology.

6 General reconstruction of $F(z)$

We consider now the reconstruction of $F(z)$ for given Hubble parameter $H(z)$ and potential U. As can be seen from (24,21), $H(z)$ is a functional of $w(z)$. We will consider several cases corresponding to phantom divide crossing at very small redshifts as favoured by the latest observations. It is desirable to derive some general properties of the behaviour of our system before embarking on the study of specific models.

It is easy to derive the following general property shared by a large class of models: if we require that $\phi'^2 > 0$ and also $U \geq 0$, then the following inequality must be satisfied

$$\frac{F'}{F_0}\left(\frac{(1+z)^2}{4}\frac{F'}{F} - (1+z)\right) + \frac{F}{F_0} > (1+z)^3 h^{-2}(z)\Omega_{m,0} \ . \tag{75}$$

Note that the r.h.s. of (75) tends to one in any model for which $w_{DE} < 0$ and it will tend to this asymptotic value quickly for $w_{DE} < -0.5$. We can consider several particular cases:

- It is seen from (75) that models which at some redshift $z_m \gg 1$ satisfy $F'(z_m) = 0$ and $0 < F(z_m) < 1$ sufficiently small, must also have $\phi'^2(z_m) < 0$, such an example is actually illustrated in Figure 1.

- When $F \to 0$ for $z \to z_m$ while at the same time $F'(z_m) \neq 0$, it follows from the expression (10) for ϕ'^2 and using (8) that $\phi'^2 \to \infty$.

- Finally, an interesting case is provided when $F(z_m) = F'(z_m) = 0$ for some z_m. We see first from (8) that $\Phi'^2 \to \Phi'^2(z_m)$, where $\Phi'^2(z_m)$ is a (small) negative number. Let us assume that $\frac{F'}{F}$ is bounded when $z \to z_m$. In that case the inequality (75) cannot be satisfied, hence $\phi'^2(z_m) < 0$ (allways assuming $U > 0$) and actually we have $\phi'^2 \to -\infty$ for $z \to z_m$ as can be checked directly with the definition of ϕ'^2. As a consequence, if $\phi'^2 > 0$ as $z \to z_m$ then we must have $\left|\frac{F'}{F}\right| \to \infty$ and if in addition $\frac{F'^2}{F}$ is bounded, then $\phi'^2 \to \infty$.

Some of these properties are illustrated with Figures 1,2,3.

Let us consider now the space-time background evolution encoded in the quantity $h(z)$. From a theoretical point of view, as is seen from (24), a given function $w(z)$ implies a corresponding functional form $h(z)$, and conversely using (22). Future data are expected to mesure $D_L(z) = H_0^{-1} d_L(z)$, and therefore $h(z) = H_0^{-1}\ H(z)$, with high precision. In the meantime, we can try some particular expressions $w(z)$, or $h(z)$, parametrized with the help of a limited number of free parameters, an attitude which turns out to be very fruitful. We will consider both constant and variable equation

of state parameter w. Observations suggest that w can be varying and we will model this variation using the following two-dimensional parametrization of the equation of state parameter suggested in [17], [18])

$$w(z) = (-1+\alpha) + \beta\ (1-x) \equiv w_0 + w_1\ \frac{z}{1+z}\ . \tag{76}$$

where $x \equiv \frac{a}{a_0}$. We will sometimes compare our results with the parametrization suggested in [19]

$$\epsilon(z) = A_0 + A_1\ (1+z) + A_2\ (1+z)^2\ . \tag{77}$$

By definition, $\epsilon(0) = 1$, hence $A_0 + A_1 + A_2 = 1$. Note that we have for (76)

$$\epsilon(z) = (1+z)^{3(\alpha+\beta)} \mathrm{e}^{-3\beta \frac{z}{1+z}}\ . \tag{78}$$

Actually, rather than delimiting some restricted bounded domain in the parameter space, some observations on small redshifts $z \lesssim 0.35$ single out a prefered direction. Variations along this direction are essentially unconstrained while variations normal to it are most efficiently constrained. This is the case for baryon oscillations data which constrain $h(z)$ and therefore $w(z)$ as follows

$$\Omega_{m,0}{}^{1/2} h(z_1)^{-1/3} \left[\frac{1}{z_1} \int_0^{z_1} \frac{dz}{h(z)} \right]^{2/3} \leq 0.469 \pm 0.017\ , \tag{79}$$

with $z_1 = 0.35,\ \Omega_{m,0} = 0.3$.

When we use the parametrization (76), eq.(79) translates into the constraint at the 1-σ level in the parameter plane $\alpha,\ \beta$ (equivalently $w_0,\ w_1$),

$$\alpha + 0.112\ \beta = 1 + w_{DE,0} + 0.112\ w_1 = 0.23 \pm 0.20\ . \tag{80}$$

As said in the Introduction, we see that these constraints allow for a w_0 close to (and possibly slightly lower than) -1 if we take a constant equation of state, while a w_0 significantly lower than -1 is allowed but it requires a steeply increasing w near $z = 0$ in agreement with recent analysis of the data. It is natural to use the constraint (80) when we consider the small z expansion of our scalar-tensor model quantities and of our fit (76).

7 Reconstruction for a constant potential U

Setting $U = 0$, we consider either $w_{DE} < 0$ constant, or else $w_{DE}(z)$ of the form (76) or (77), whose variation is parameterized by two parameters. This extends the investigations in [49], where only the case $w_{DE} = -1$ was considered. The interesting result obtained there was that F vanishes, and hence the theory is singular, at very low redshifts $z \approx 0.66$ for $F'(0) = 0$ and at only slightly higher redshift $z \approx 0.68$ if one fully exploits the possible initial conditions allowed by the solar system constraints. It is therefore interesting to investigate how this result is affected when we take different equations of state. In particular we will also consider here equations of state with

$w_{DE} < -1$. As SNIa observations point to a varying equation of state which is of the phantom type on very low redshifts $z \lesssim 0.3$, this case is considered too. For the polynomial expression (77), for definiteness we will use the best fit to the "Gold data set" consisting of 156 supernova which was proposed in [27] (with the priors $\Omega_{m,0} = 0.3$ *and* $\Omega_{k,0} = 0$):

$$A_1 = -5.94 \pm 3.61 \qquad A_2 = 2.39 \pm 1.47 \ . \tag{81}$$

These values are similar to those earlier obtained in [20] for the same dataset. As we can see, (81) has large uncertainties. The corresponding equation of state is today of the phantom type but rapidly increases to a positive value $w_{DE} \simeq 0.1$ at $z \simeq 0.8$ and then decreases to its asymptotic value $w_{DE} = -\frac{1}{3}$ but it is still slightly positive at $z = 2$. We first note from (54) that F_2 satisfies

$$F_2 = \frac{3}{2}\ (w_{DE,0} - 1)\ \Omega_{DE,0} + 3\ \Omega_{U,0}\ . \tag{82}$$

Therefore $F_2 < 0$ for $U = 0$ as long as $w_{DE,0} < 1$ while F_2 decreases with decreasing $w_{DE,0}$. The results obtained numerically are shown in Figure 4. It is seen that we keep essentially the same picture, in particular F vanishes more rapidly when w_{DE} decreases, hence the problem is even more severe for phantom DE with constant w_{DE}. Even the fit (81) where w_{DE} varies substantially at low redshifts yields essentially the same behaviour. The same result as for (81) is obtained using the parametrization (76) with $\alpha = -0.377$ and $\beta = 2$. This is similar to earlier results showing that a large variation of the equation of state around $w_{DE} = -1$ on low redshifts can result in essentially the same $d_L(z)$ [17, 33] for $z \lesssim 1$ and one can understand from the general expression eq.(31) why all cases displayed in Figure 1. will have basically the same behaviour regarding the evolution of $F(z)$. The initial condition $F'_0 = F_1 = 0$ and $\Phi'(0) \neq 0$ means physically that the theory corresponds strictly to General Relativity today, $\omega_{BD,0}^{-1} = 0$. The solar system constraints allow for a very small nonvanishing F_1, see eq.(15), and the corresponding change in z_{max} is very marginal as can be seen on Figure 4. To summarize, looking only at $z = 0$ one could think that models with $w_{DE,0} = -1$ and $U = 0$ are allowed, however we see that the cosmological evolution of such models leads to a singularity at very low redshifts generalizing the results obtained earlier [49]. It is clear from these results that a scalar-tensor theory of gravity with vanishing potential U is definitely excluded by the Supernovae data.

The next natural step is to consider constant (nonvanishing) potentials U. Of course, one does not expect such a behaviour to be relevant up to very large redshifts, but it is certainly a sensible approximation to start with on small redshifts $z \lesssim 1.5$. We note from (82) that $F_2 < 0$ for $w_{DE,0} < -1$. By taking $\Omega_{U,0} \approx \Omega_{DE,0}$ one has the smallest possible F_2 for given $w_{DE,0}$ and these are the cases which are found to have the largest range of validity. We find that many models are allowed which are perfectly viable on small redshifts $z \lesssim 1.5$. Typically, these models become singular at some higher redshift well beyond $z = 2$. There are several possibilities: either it is the quantity F that is vanishing first or it is the quantity ϕ'^2 that vanishes first. As discussed in Section 6, when $F = 0$ and $F' \neq 0$, the quantity ϕ'^2 diverges, $\phi'^2 \to \infty$.

Models for which both F and F' vanish together can be considered as a limiting case which gives the largest possible redshift. If we change very slightly the potential U so that when $F' = 0$, $F > 0$, one gets $\phi'^2 = 0$ earlier. These different possibilities are displayed in Figures 2,3. We have checked the behaviour of such models with w_{DE} starting below -1 as favoured by SNIa data and with w_{DE} quickly becoming larger than -1. To summarize, we find that models with $\Omega_{U,0} \to 0.7$ *and* an equation of state with $w_{DE} \to -1$ will become singular at arbitrarily high redshifts. We should remember that $\Omega_{U,0} = 0.7$ together with $w_{DE} = -1$ gives back General Relativity ($F = 1$). In all other cases some maximal redshift is found where the model becomes singular.

In all our numerical calculations, we neglected radiation since its energy density is very small at redshifts of interest. However, even in principle its presence cannot prevent the occurence of the singularity at the moment when G_{eff} changes sign ($\phi'^2 = 0$ in our case) whose generic (anisotropic) structure is independent of the matter equation of state (see [55] in this respect).

8 Asymptotic stability

The next step is to consider nonconstant potentials. As we have shown very generally in Section 5, such models are consistent with DE of the phantom type today and phantom divide crossing at small redshifts. An example of the reconstruction of a model with phantom DE today and phantom divide crossing at $z \approx 0.3$ was presented in [50]. However, in this paper the reconstruction was implemented only for small redshifts $z \lesssim 2$. As was shown in the previous section, one has to consider the large-z behaviour of a model, too, to prove its viability.

In contrast to the case with constant or vanishing potential $U(\Phi)$, a growing potential $U(\Phi)$ allows for the construction of scalar-tensor DE models which are viable for all redshifts and evolves according to the fit (76)

$$w(z) = \text{const} = w_0 + w_1 = -1 + (\alpha + \beta) > -1 \qquad\qquad z \gg 1 \ . \tag{83}$$

As a particular example of such a model, let us assume that the scalar-tensor gravity approaches GR sufficiently fast at the matter-dominated stage and that DE 'tracks' matter:

$$\begin{aligned} F &\to F_\infty = \text{const}, \quad |\dot{F}| \ll H F_\infty, \quad |\ddot{F}| \ll H^2 F_\infty, & (84)\\ H^2 &\propto (1+z)^3 \qquad\qquad 1 \ll z \ll z_{eq} \ , & (85)\end{aligned}$$

where z_{eq} is the redshift at the matter-radiation equality. In this way we recover in the past the usual behaviour $a \propto t^{\frac{2}{3}}$. However, the constant value $F_\infty < F_0$ may not be too small compared to F_0. In order to satisfy the BBN constraints [69, 70], the following inequality is required

$$(F_0 - F_\infty)/F_0 < 0.1 \ , \tag{86}$$

which can be easily satisfied. Indeed, as can be seen from (75), we have for constant F and $U > 0,\ \phi'^2 > 0$

$$F_0 - F_\infty < \frac{\rho_{DE}}{\rho_m + \rho_{DE}} F_0 \ . \tag{87}$$

On the other hand, the assumption $F_0 > F_\infty$ provides a good matching to the small-z expansion derived in Sec. 5 with $F_2 < 0$ and a very small F_1. In view of (86), this matching should occur at a sufficiently small z, too.

This scaling behaviour of DE corresponds to the asymptotic solution (85) with $w_0 = -w_1$, equivalently $\alpha + \beta = 1$. As is well known, it can be obtained by taking an exponential potential $U(\Phi)$ at $\Phi \to \infty$, i.e. at large redshifts:

$$U \propto \mathrm{e}^{\sqrt{\frac{3}{2F_0\Omega_{U,\infty}}}\Phi} \ , \tag{88}$$

where $\Omega_{U,\infty}$ is the constant asymtotic value of Ω_U at $z \gg 1$ which is a free parameter formally. However, actually it should be small (less than a few percents) to obtain the correct value of the growth factor of density perturbations during the total matter-dominated stage. The total DE energy density in terms of the critical one $3F_0H^2$ is

$$\Omega_{DE,\infty} = 2\Omega_{U,\infty} + \frac{F_0 - F_\infty}{F_0} < 1 \ . \tag{89}$$

Note that the term $\frac{F_0 - F_\infty}{F_0}$ in (89) always has the same equation of state as the main background matter. Thus, to obtain behaviour different from $w_0 = -w_1$ at large z during the matter dominated stage is possible if $F_0 = F_\infty$ only which requires additional fine tuning and is not natural.

9 Conclusions

In this work we have considered the viability of scalar-tensor models of Dark Energy. We have used different types of observations: Solar System constraints which constrain the model *today*, and other data like the supernova data which constrain its cosmological evolution, in particular the time evolution of the DE equation of state parameter w_{DE}. We were interested specifically in models which violate the weak energy condition on small redshifts $z \lesssim 0.3$ and in any case today. We have found the formal general integral solution for $F(z)$ when we reconstruct it for given $H(z)$, which can be obtained from the $d_L(z)$ data, and for given $U(z)$. This general solution allows immediately for an integral representation of scaling solutions. We have constructed scaling solutions and shown that they exist in models with $F = \alpha\Phi^2$ with α =constant, in these models the Brans-Dicke parameter ω_{BD} is constant. Only for nonzero potentials can these models have scaling solutions with constant $w_{DE} < -1$. However, it is shown that for these models $|w_{DE} + 1|$ is very small with $|1 + w_{DE}| \sim \omega_{BD}^{-1}$. We have further performed systematically the small z expansion of the theory and used it to extract various observational constraints. We recover that a large positive $\omega_{BD,0}$ requires $|F_1| \ll 1$ where F_1 is the first derivative today of $\frac{F}{F_0}$. We find that a significantly phantom DE today ($w_0 < -1.01$) implies that F_2, the second derivative

today of $\frac{F}{F_0}$, must be negative and not small, thus, significantly larger than F_1 by modulus. However, while necessary this is not a sufficient condition for phantom DE today, for example for vanishing potential $F_2 < 0$ whenever $w_{DE,0} < 1$.

The condition $|F_2| \sim 1$ while the Solar System data require $|F_1| < 10^{-2}$ (i.e. anomalously small) is the only 'fine tuning' required to get a significantly phantom DE at the present time in scalar-tensor gravity. Note that, since the derivatives $F_i(z)$ are not parameters of the effective microscopic Lagrangian (1) but depend on initial conditions in the early Universe too, it could be even better to call this a "cosmic coincidence". Our point of view is that the condition $|F_2| \sim 1$ cannot be excluded by pure thought, only observations will possibly do it: in the absence of this condition when all F_i are of the same order as F_1, the general prediction is that the amount of possible phantomness in scalar-tensor DE models is very small, less than 1%.

As for the solar system constraints, we have shown that the Post-Newtonian parameter $\gamma_{PN,0}$ must satisfy $\gamma_{PN,0} - 1 \simeq -\omega_{BD,0}^{-1} < 0$, this is a general requirement for the viability of our model. On the other hand, a significantly phantom DE today implies $\beta_{PN,0} > 1$. Combining those results, we find that the negative quantity $\frac{\gamma_{PN,0}-1}{\beta_{PN,0}-1}$ does not depend on F_1 and can be expressed in function of F_2 and $1+w_{DE,0}$. This would enable us to find F_2 from the Solar System constraints provided we know $1 + w_{DE,0}$ from other cosmological data. Still for phantom DE today, we find that $\dot{G}_{\rm eff,0}$ has the same sign as F_1. So, a measurement of the sign of $\dot{G}_{\rm eff,0}$ would give us the sign of F_1 if we know we have $1 + w_{DE,0} < 0$. On the other hand, a measurement of both quantities with opposite signs would rule out phantom DE today though measuring the sign of F_1 can be hard to determine observationally. However, due to the already confirmed smallness of F_1, connection between cosmological and Solar System tests of dark energy is rather one-way since it requires much greater accuracy from the latter ones for the determination whether DE is phantom at present or not. While a positive detection of $\beta_{PN,0} > 1$ is a strong argument for the phantom DE at present, a negative result (no measurable deviation of $\beta_{PN,0}$ from unity) tells us nothing regarding DE properties. Also, the Solar System tests are clearly unable to provide w_1 in any reasonable future, so no information about the possibility of the phantom divide crossing may be expected from them.

We have also considered numerically the reconstruction of F for various w_{DE}, including constant as well as varying equations of state of the phantom type. Generalizing results obtained in [49] for a pure cosmological constant, we find that models with a vanishing potential (see Figure 3) are ruled out and lead to a singular behaviour for $z < 0.66$. While it is clearly possible to have phantom DE today with $U = 0$ without conflicting with the data, the cosmological evolution of these models rules them out. For models with constant nonvanishing potentials, which can be considered as a good approximation on small redshifts for more general models with varying potentials, we find that it is easy to have models in agreement with observations on small redshifts $z \lesssim 2$. However, it is interesting that these models generically have a maximal redshift where they become singular. So, to construct a scalar-tensor DE model having a sufficiently long matter-dominated stage, a non-constant potential $U(\Phi)$ is required, and we have presented an example of such a model where DE

tracks matter at large redshifts.

Therefore, the final conclusion is that the generic scalar-tensor gravity (1) with the two functions $F(\Phi)$ and $U(\Phi)$, derived from some underlying theory (e.g., from brane cosmology) or taken from observational data, has enough power to provide internally consistent cosmological models with a temporarily phantom DE (at present, in particular) and with a regular phantom divide crossing in the course of the evolution of the Universe.

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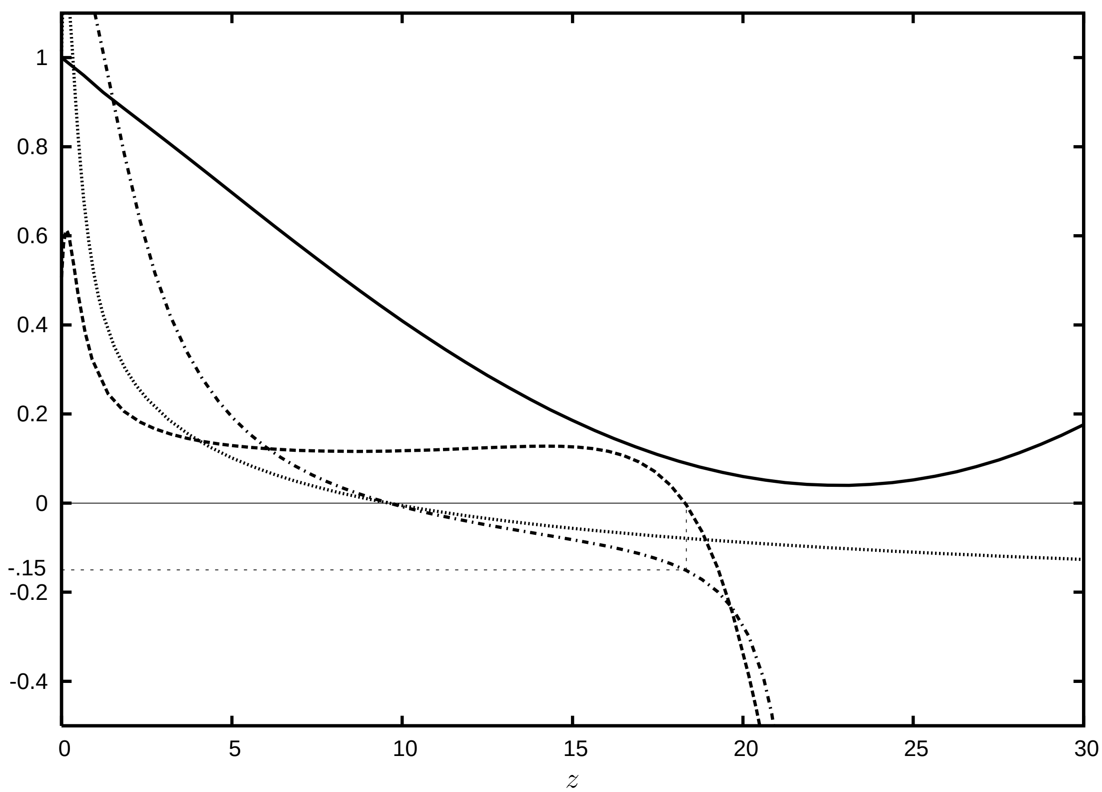

Figure 1: Several quantities are displayed for the model with parametrization (76) with $\alpha \equiv 1 + w_0 = -0.2$ and $\beta \equiv w_1 = 0.4$, while $F_1 = 0$ and $\frac{\Omega_{U,0}}{\Omega_{DE,0}} = \frac{\Omega_{U,0}}{0.7} = 0.97585$. The curves shown represent the following (rescaled) quantities in function of redshift z: $\frac{F}{F_0}$ (solid), $10 \times \Phi'^2$ (dotted), $10 \times \phi'^2$ (dashed) and $0.1 \times \omega_{BD}$ (dot-dashed). It is seen that ω_{BD} and Φ'^2 become negative at $z \approx 10$. The model remains valid beyond $z \approx 10$ untill $z \approx 18$ as long as $\omega_{BD} > -\frac{3}{2}$ or equivalently $\phi'^2 > 0$. So, we see that the model remains valid for a large interval where $\Phi'^2 < 0$. Of course, it is impossible to reconstruct Φ in this interval using the $Z = 1$ parametrization. Note that ϕ'^2 becomes negative *before* $z_m \approx 22$ where $F'(z_m) = 0$, in accordance with (75).

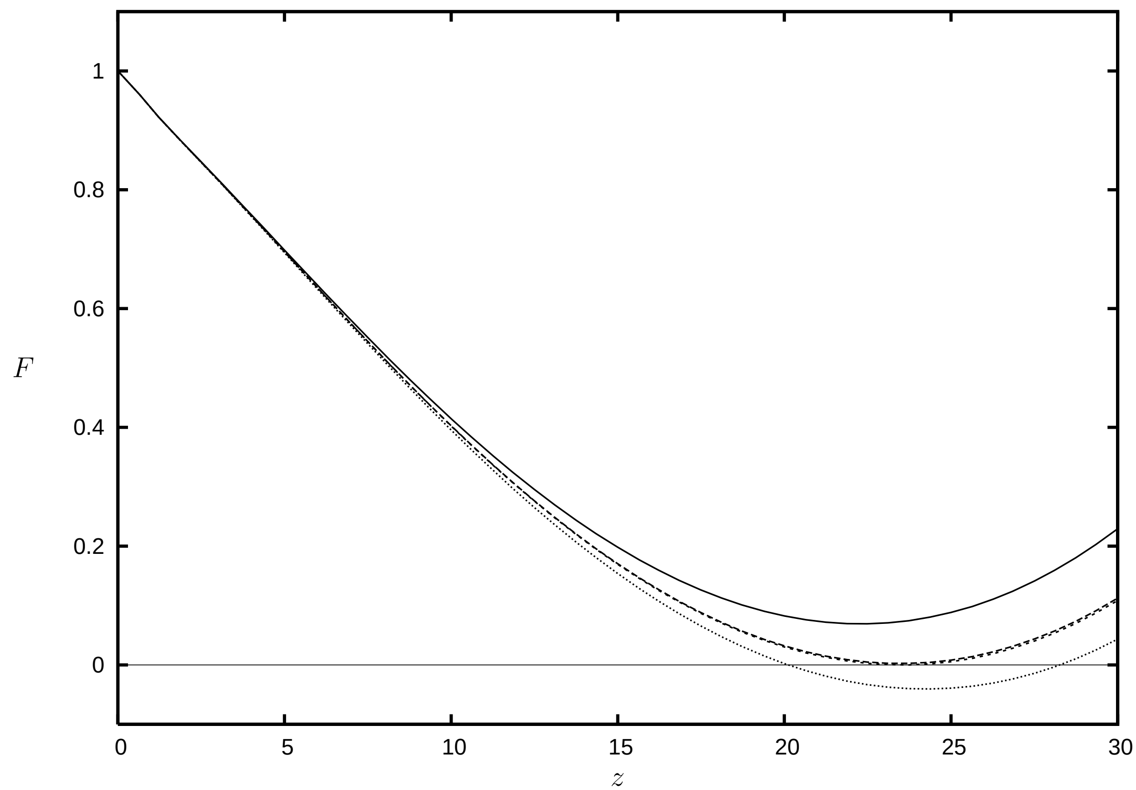

Figure 2: The quantity $\frac{F(z)}{F_0}$ is displayed for the parametrization (76) with $\alpha \equiv 1 + w_0 = -0.2$ and $\beta \equiv w_1 = 0.4$ and $F_1 = 0$. We have the following values for $\frac{\Omega_{U,0}}{\Omega_{DE,0}} = \frac{\Omega_{U,0}}{0.7}$ from bottom to top: 0.9758, 0.975824492, 0.975826, 0.97587. The second curve has its minimum at $F = 0$ and is superimposed on the third curve which has its minimum at $F = 2.4 \times 10^{-3}$.

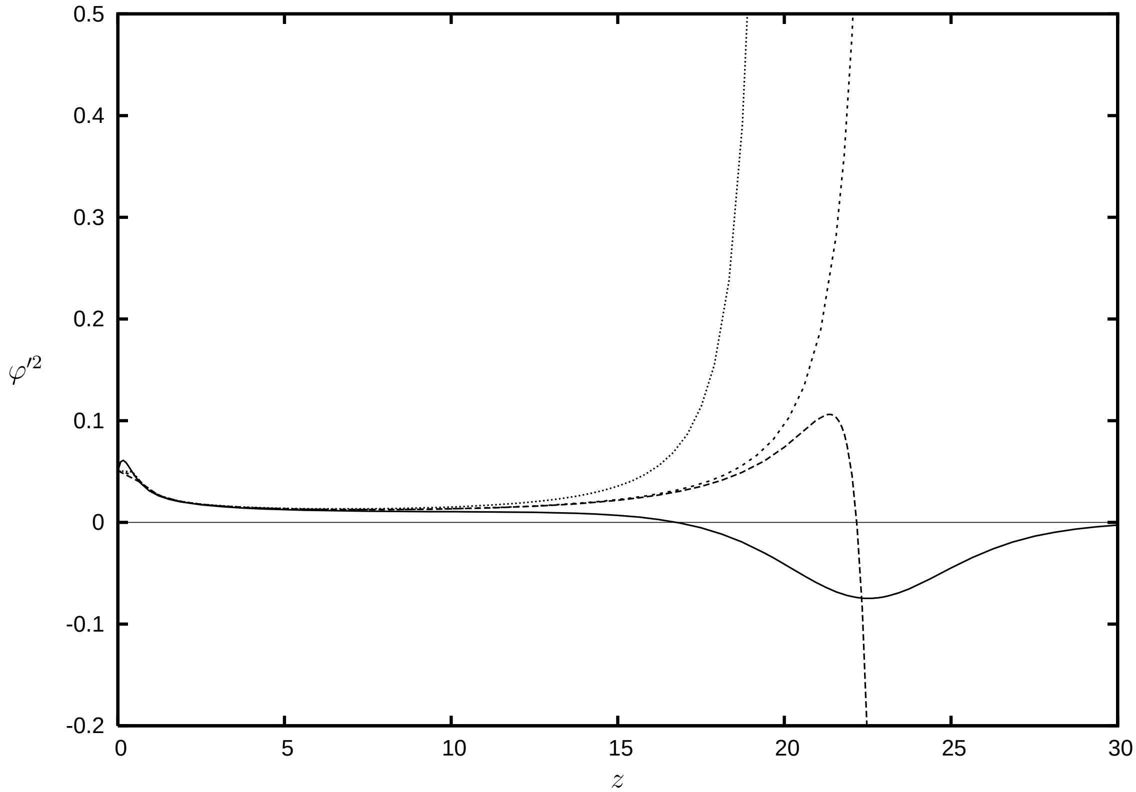

Figure 3: The quantity ϕ'^2 is shown for the same models as Figure 2. The short, resp. long, dashed curve corresponds to $\frac{\Omega_{U,0}}{\Omega_{DE,0}} = \frac{\Omega_{U,0}}{0.7} = 0.975824492,$ resp 0.975826.

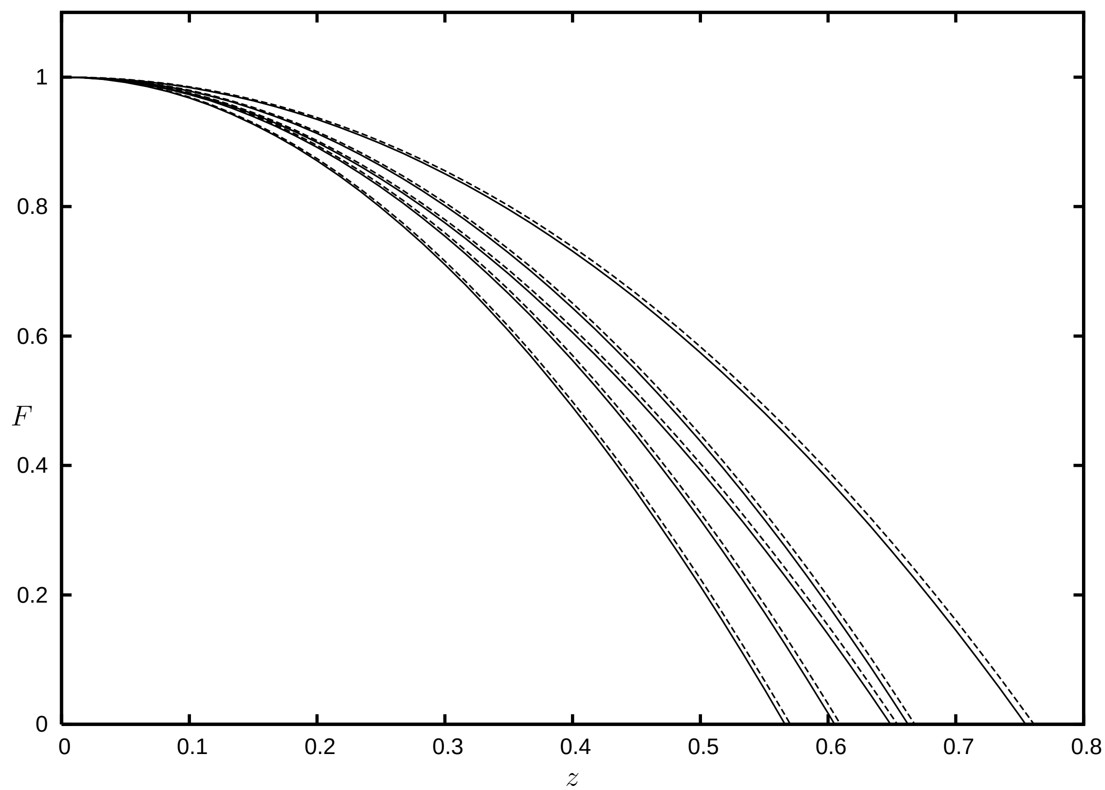

Figure 4: The function $\frac{F}{F_0}$ is shown for several models with vanishing potential $U = 0$. The solid lines correspond to the initial conditions $F_0' = 0$ while the dashed lines correspond to the maximal values allowed by the solar system constraint $\omega_{BD,0} > 4 \times 10^4$. We have from left to right the following equation of state parameter w: $-2, -1.5$, polynomial expression (81), $-1, -0.5$. It is seen that the limit of regularity of these models corresponds to very low redshifts, $0.566 \leq z \leq 0.663$ for $-2 \leq w \leq -1$. Note that the polynomial expression (81) represents crossing of the phantom divide.

APPENDIX

We summarize in the table below, the integrability of scaling solutions using (31) for several exponents $n \equiv 3\gamma$. We use the following abbreviations:
a.n.i.: analytically non integrable
Elliptic, log, Hypergeo : solutions expressed in terms of resp., elliptic, log (or $\arg\tanh$) and hypergeometric functions
$P^{(n)}(x)$: polynomial in x of degree n

n w	< 0 < -1	0 -1	1 $-2/3$	2 $-1/3$	3 0	4 $1/3$	5 $2/3$	6 1	7, 8, 9 $2/3, 5/3, 2$
A_0	a.n.i.	Elliptic	Elliptic	log	$-\frac{2}{\sqrt{(1+k)x}}$	$\frac{P^{(1)}(x)}{x\sqrt{k+1/x}}$	Elliptic	Elliptic	Hypergeo
A_1	a.n.i.	log	Elliptic	log	$-\frac{2}{3\sqrt{(1+k)x^3}}$	$\frac{P^{(2)}(x)}{x^2\sqrt{k+1/x}}$	Elliptic	$-\frac{2}{3}\frac{1+kx^3}{\sqrt{x^3+kx^6}}$	Hypergeo
A_4	a.n.i.	log	Elliptic	log	$-\frac{2}{9\sqrt{(1+)x^9}}$	$\frac{P^{(5)}(x)}{x^5\sqrt{k+1/x}}$	Elliptic	$\frac{2}{9}\frac{(1+kx^3)(-1+2kx^3}{x^3\sqrt{x^3+kx^6}}$	Hypergeo
β_1	a.n.i.	a.n.i.	a.n.i.	a.n.i.	$\frac{1}{3(1+k)x^2}$	$\frac{1}{3x^2} - \frac{4k}{3x}$	a.n.i.	a.n.i.	a.n.i.
β_2	a.n.i.	a.n.i.	a.n.i.	a.n.i.	$\frac{2}{45(1+k)x^5}$	$\frac{2}{45x^5} - \ldots + \frac{256k^4}{314x}$	a.n.i.	a.n.i.	a.n.i.

where $$A_n \propto \int \frac{\mathrm{d}x}{x^n h(x)} \qquad \beta_1 \propto \int \frac{\mathrm{d}x}{h(x)} \int^x \frac{\mathrm{d}\eta}{\eta h(\eta)} \qquad \beta_2 \propto \int \frac{\mathrm{d}x}{h(x)} \int^x \frac{\mathrm{d}\eta}{\eta^4 h(\eta)}$$